Indian Statistical Institute
Semestral Examination
Differential Geometry I
MMath I

Max Marks: 60 Time: 3 hours

Answer all questions.

(1) (a) Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Show that its curvature κ is
$$\kappa = \frac{||\ddot{\gamma} \times \dot{\gamma}||}{||\dot{\gamma}||^2}$$
where $\dot{\gamma} = d\gamma/dt$. [8]

(b) Define the term *torsion*. Compute the curvature and torsion of the circular helix
$$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$$
where $-\infty < \theta < \infty$. [8]

(c) Describe all curves in $\mathbb{R}^3$ which have constant curvature $\kappa > 0$ and constant torsion τ. Justify. [4]

(2) (a) Give the definitions of: (i) length of a curve on a surface, (ii) isometry between surfaces. Is the map from the cone $x^2 + y^2 = z^2$, $z > 0$, to the plane given by $(x, y, z) \mapsto (x, y, 0)$ an isometry? [6]

(b) When is a map between surfaces conformal? Let $f(x)$ be a smooth function. Let $\sigma(u, v) = (u\cos v, u\sin v, f(u))$ be the parametrization of the surface of revolution S obtained by rotating the curve $z = f(x)$ in the xz-plane about the z-axis. Find all functions f for which σ is conformal. [8]

(c) Define the normal and geodesic curvatures of a unit speed curve γ on a surface S. Show that the normal curvature of any curve on a sphere of radius r is $\pm 1/r$. [6]

(3) (a) Discuss how the principal curvatures at a point on a surface are defined. Compute the principal curvatures at $p \in S$ in the cases where (i) S is the sphere of radius 1, (ii) $S = S^1 \times \mathbb{R}$ where S^1 is the unite circle in the xy-plane. [6]

(b) Let $\sigma(u, v) = (u + v, u - v, uv)$ be a parametrization of a surface S. Calculate the Gaussian and mean curvatures at the point $(2, 0, 1)$. [8]

(c) Define the term *geodesic*. Show that an isometry between surfaces takes geodesics to geodesics. [6]